Network Mathematics - Why is it a Small World?

Oskar Sandberg

Networks

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- This is an abstraction which can be used to describe a lot of different systems (technical, physical, biological, sociological, etc. etc.).

Networks

Math	**Graph**	**Vertex**	**Edge**
CS	Network	Node	Link
Physics	System	Site	Bond
Sociology	Social Network	Actor	Tie
		Individual	Friendship
	WWW	Webpage	Link (d)
	Internet	Site	Connection
		Network	Bridge
	Road System	Crossing	Road

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- Randomly evolved: The Web, social networks.
- Somewhere in between: The Internet, P2P Networks.

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- When designing structured networks, questions are usually algorithmic. (How do I create a network with this property?)
- When studying randomly generated networks questions tend to analytic. (Does the network have this property?)

Random Graph Theory

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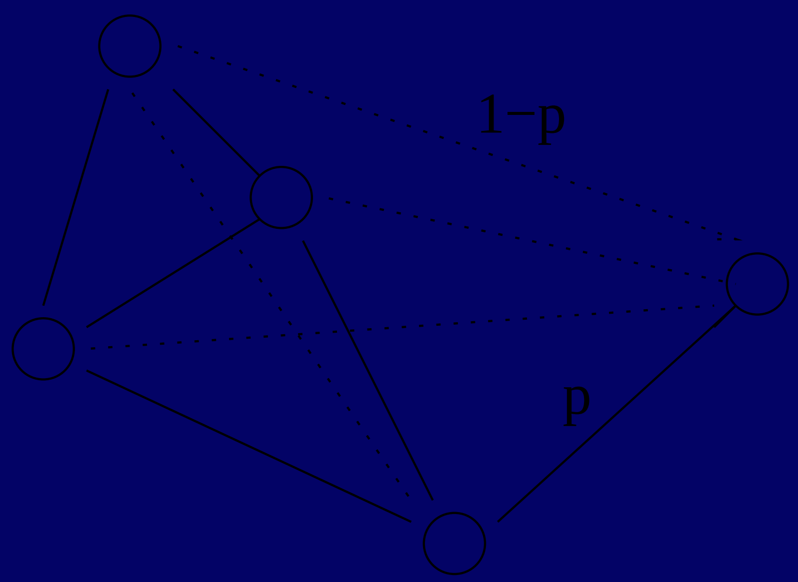
The simplest model for a random graph $G(n,p) = (V,E)$:

- $V = \{0,1,2,\ldots,n\}$
- $u \leftrightarrow v$ (that is $(u,v) \in E$) independently and with probability p for every pair of vertices u and v.



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- If $p > 1/n$ “most” of the vertices form one connected cluster.
- If $p > \log n/n$ all of the vertices are connected.
- The “diameter” of the connected cluster is $\log n$.

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In recent years, new models have been introduced for networks with various properties.

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- It was famously illustrated for social networks by Stanley Milgram in 1967.
- He experimented by having volunteers in Omaha, Nebraska forward letters to a stockbroker in Boston through friends.
- Milgram reported that on average the packages reached their destination in only six steps.

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From:

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To:

Oskar Sandberg, Göteborg, Sweden.

An Example. cont.

Fernanda
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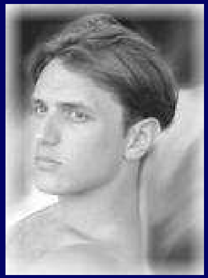
An Example. cont.

Fernanda Trincado

Knows:



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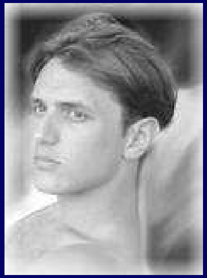

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Is related to:

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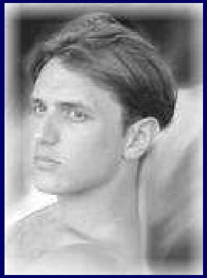

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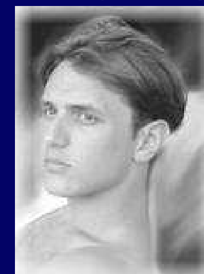

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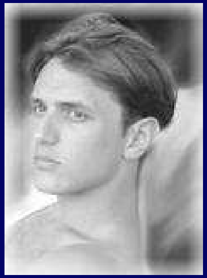

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- As noted, however, they are not a good model for social networks.
- It isn't possible to *search* in them.

Kleinberg's Model

- For searching to be possible, vertices need to have locations, and whether $u \leftrightarrow v$ should depend on the distance between them ($d(u,v)$).

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- α tunes the degree of "locality" the shortcuts.
- Route using *greedy routing*: step to the neighbor which is closest to destination.

Kleinberg's Model, cont.

Efficient routing is possible when α is such that:

$$\mathbf{P}(x \rightsquigarrow w) \propto \frac{1}{\#\text{ nodes closer to } x \text{ than } w}$$

This can be seen to be $\alpha = d$, where d is the dimension of the space (2 in the simulations).

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- Kleinberg’s result is mostly negative: for the vast majority of networks, searching is not possible.
- Why should one expect real-world networks to have the necessary edge distribution?

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- Consider only the relative size of the first k numbers drawn.
- These have a random order: each is equally likely to be the biggest of them.
- Thus the k-th number has probability $1/k$ of being the biggest one yet.

Interest Model

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- Let u associate with each other node v a random quantity representing u's interest in v.
- Let $u \leftrightarrow v$ if u is more interesting to v than any node which is closer.

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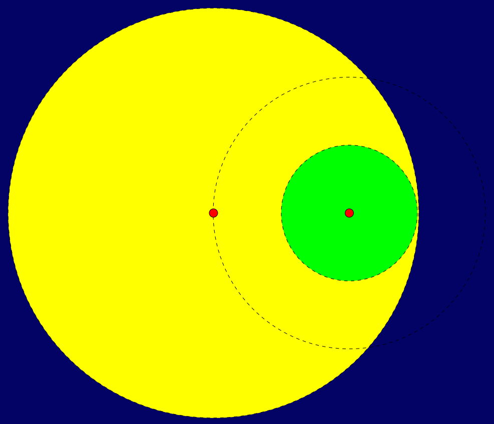
  This is now independent for each v.
- Expected number of shortcuts from each node is $\log n$.
- One can see that greedy routing takes $O(\log n)$ steps on a graph generated like this.

A Proof



- If d is the distance between u and v, the yellow disk is the vertices within $(3/2)d$ of u and the green within $d/2$ of v.

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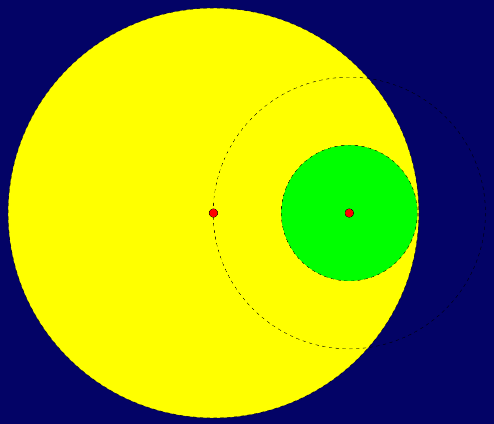

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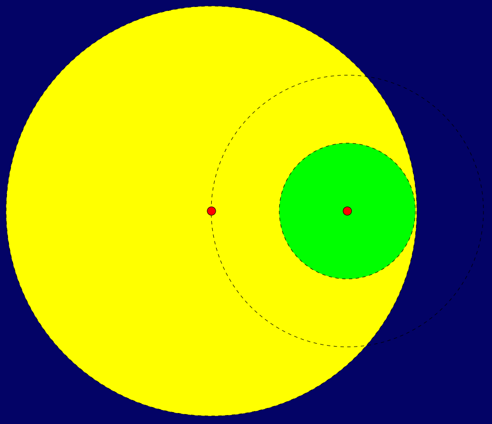

- If d is the distance between u and v, the yellow disk is the vertices within $(3/2)d$ of u and the green within $d/2$ of v.
- u must have a shortcut to the very “most interesting” vertex in the yellow disk.
- The probability that that vertex is in the green part is $1/9$.

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- Let u's interest in v be the inverse of $|p(u) - p(v)|$.
- That is: $u \leftrightarrow v$ if $p(u)$ is closer to $p(v)$ than p of any node closer to u to than v.

The Double Clustering Graph

Definition 1 *Let* $(x_i)_{i=1}^n$ *and* $(y_i)_{i=1}^n$ *be two sequences of points without repetition in possibly different spaces* M_1 *and* M_2 *with distance functions* d_1 *and* d_2 *respectively. The digraph* $G = (V, E)$ *is constructed as follows:*

- $V = \{1, 2, \ldots, n\}$.
- $(i, j) \in E$ *if for all* $k \in V$, $k \neq i, j$:

$$d_1(x_i, x_k) < d_1(x_i, x_j) \Rightarrow d_2(y_i, y_k) \geq d_2(y_i, y_j)$$

(Make undirected by removing directionality of the edges.)